ENDOMORPHISMS AND DIRECT DECOMPOSITIONS IN LATTICES

BY

LOIS AILEEN HOSTINSKY

B.S., Kansas State College, 1943
A.M., University of Illinois, 1945

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
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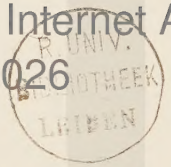

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ENDOMORPHISMS AND DIRECT DECOMPOSITIONS IN LATTICES*

Introduction

The object of this investigation is to obtain a generalization of results in the decomposition theory developed by Baer for operator loops[1]. The two problems treated here relate to criteria for uniform splitting and to refinement theorems. In view of our purposes, the algebraic system considered is a complete modular lattice which satisfies one additional property. Although the splitting criteria found are of a nature somewhat different from those in loop theory, analogous refinement theorems have been obtained.

In our study of decomposition theory we make use of a generalization of Fitting's method[2], which is based upon the fact that direct decompositions are defined by their decomposition endomorphisms, and employ products of these decomposition endomorphisms as our chief tool. Moreover, we work entirely within the indicated partially ordered set. Thus the elements (of an original domain) are completely eliminated and not available for use in proofs or in the concept of homomorphism, for example.

I. Fundamental Concepts

We consider a complete modular lattice P, consisting of elements denoted by lower-case Latin letters, for which sums and products exist, and impose upon P the following additional postulate: If a and $\{ p_\nu \}$, where ν ranges over a set of ordinals, are elements in P and if the p_ν form an ascending chain,

*The author wishes to express her appreciation to Professor Reinhold Baer, who directed the preparation of the thesis of which this is an abstract.

1. R. Baer, **The Homomorphism Theorems for Loops**, Amer. J. of Math., vol. 67 (1945), pp. 450-460.
   R. Baer, **Splitting Endomorphisms**, Trans. A. M. S., vol. 61 (1947), pp. 508-516.
   R. Baer, **Direct Decompositions**, Trans. A. M. S., vol. 62 (1947), pp. 62-98.
2. N. Jacobson, **The Theory of Rings**, Mathematical Surveys, vol. 2, New York, 1943.

then $a \Sigma_\nu p_\nu = \Sigma_\nu a p_\nu$.

Without loss in generality we postulate the existence of an element 0 in P such that $0 \leq p$ for every p in P. The system of all the normal and admissible subloops of an operator loop or normal and admissible subgroups of an operator group satisfies all of the postulates and provides the most appropriate example for our purposes.

A subset of P is called a substructure if it is closed with respect to the formation of sums and products. Any two elements p and q of P such that $q \leq p$ define a substructure consisting of all elements r such that $q \leq r \leq p$. Such a substructure is called a quotient, denoted by p/q, and satisfies all of the postulates.

If p is an element in P, if q is an element in P or is a system satisfying the same postulates as P, and if there is given a single-valued transformation η whereby to every $x \leq p$ there corresponds $x\eta \leq q$, then η is said to be a *mapping* of p *into* q. The image of x under η is denoted by $x\eta$, and $p\eta$ may be a proper part of q. If $p\eta = q$, η is termed a mapping of p *upon* q. The mapping is *exhaustive* if η maps every $x \leq p$ upon $x\eta \leq q$ so that if $z \leq x\eta$ then there exists an element $z' \leq x$ such that $z'\eta = z$.

A mapping η is an *order-preserving* mapping if $x \leq y$ implies that $x\eta \leq y\eta$ and is an *additive* mapping if $(\Sigma_\nu a_\nu)\eta = \Sigma_\nu (a_\nu \eta)$.

Definition 1.1. A *homomorphism* η of p upon q is an exhaustive and additive mapping of p upon q in which $x\eta = y\eta$ implies the existence of elements x′ and y′ such that $x'\eta = y'\eta = 0$ and $x + x' = y + y'$.

Definition 1.2. The *kernel* $k(\eta)$ of a mapping η is the sum of all elements which η maps upon zero.

It is easily seen that a homomorphism η is an additive and exhaustive mapping in which $x\eta = y\eta$ if and only if $x + k(\eta) = y + k(\eta)$.

An *isomorphism* of p upon q is a one-to-one, exhaustive, and order-preserving mapping of p upon q, and an *automorphism* is an isomorphism of p upon itself.

From these basic definitions and their simple consequences, the usual homomorphism and isomorphism theorems are derived.

Theorem 1.1. The following statements are equivalent:

1) η is additive and exhaustive and η induces an isomorphism of $p/k(\eta)$ upon q.

2) η is a homomorphism of p upon q.

3) η is additive and exhaustive and η induces an isomorphism of $x/xk(\eta)$ upon $x\eta$ for every $x \leq p$.

Theorem 1.2. If $x \leq p$, $y \leq q$, and $x\eta \leq y$, then the homomorphism η of p upon q induces a homomorphism of p/x upon q/y, defined by mapping a, for $x \leq a \leq p$, upon $a\eta + y$.

Theorem 1.3. If $a \leq p$ and $s \leq p$, then there exists an isomorphism of $(s + a)/a$ upon s/as.

Definition 1.3. The mapping of η is said to be an *endomorphism* of p if η is a homomorphism of p upon $p\eta \leq p$.

Definition 1.4. If η is an endomorphism of p and if $x\eta \leq x \leq p$, x is an η-*admissible* part of p.

Definition 1.5. The radical of the endomorphism η, denoted by $r(\eta)$, is the sum $\Sigma_i k(\eta^i)$, where $k(\eta^i)$ is the kernel of the endomorphism η^i and where the summation ranges from one to infinity.

Theorem 1.4. An endomorphism η of p induces

1) an endomorphism in $r(\eta)$.

2) an isomorphism of $p/r(\eta)$ into itself.

II. Criteria For Uniform Splitting

Definition 2.1. The element c is a *complement* of the

endomorphism η of p if $p = r(\eta) + c$, $cr(\eta) = 0$, and $c\eta = c$.

These first two conditions state that p is equal to the direct sum $r(\eta) \oplus c$.

Definition 2.2. The endomorphism η of p is said to be a *splitting endomorphism* of p or to *split* p if there exists a complement of η.

Definition 2.3. The endomorphism η of p is a *uniformly splitting endomorphism* of p if η induces a splitting endomorphism in every η-admissible element included in p.

The main results of this section are established by the introduction of the following concept.

Definition 2.4. The element x is said to be *η-automorphic* if x is η-admissible and if $y\eta = y$ for every y which is an η-admissible part of x.

Theorem 2.1. If the endomorphism η of p splits p uniformly, then the sum of η-automorphic elements is η-automorphic.

Theorem 2.2. If the endomorphism η of p splits p uniformly, then the complement $c(\eta)$ of η is uniquely determined, η-automorphic, and equal to the sum of η-automorphic elements.

Corollary. If η splits p uniformly with complement $c(\eta)$ and if x is an η-admissible part of p, then $xc(\eta)$ is the uniquely determined complement of the endomorphism which η induces in x.

These results as well as the following generalization of Fitting's Lemma[1] are of fundamental importance in the proofs of later theorems.

Theorem 2.3. If η is an endomorphism of p and if for every $x \leq p$ such that $x\eta \leq x$ there exists a positive integer $h = h(x)$ such that $x\eta^h = x\eta^{h+1} = \ldots$ and such that $xk(\eta^h) = xk(\eta^{h+1}) = \ldots = xr(\eta)$, then $x = xr(\eta) \oplus x\eta^h$, $x\eta^h$ is a com-

1. N. Jacobson, The Theory of Rings, Mathematical Surveys, vol. 2, New York, 1943.

plement of the endomorphism induced in x, and η is uniformly splitting.

Theorem 2.4. If the ascending chain condition holds for parts of p, then an endomorphism η of p splits p uniformly if and only if, for every η-admissible part x of p, there exists a positive integer n such that $x\eta^n$ is η-automorphic.

Theorem 2.5. If η is an endomorphism of p, the following conditions are equivalent:

1) η splits p uniformly.

2) a) The sum $c(\eta)$ of η-automorphic elements is η-automorphic;
   b) If $y\eta \leq y \leq p$ and $yc(\eta) \leq x \leq p$, then $y \leq x + r(\eta)$.

3) a) The sum $c(\eta)$ of η-automorphic elements is η-automorphic;
   c) If $z\eta \leq z \leq p$, then $z \leq zc(\eta) + r(\eta)$.

4) a) The sum $c(\eta)$ of η-automorphic elements is η-automorphic;
   d) If $z\eta \leq z \leq p$, then $z + r(\eta) = zc(\eta) + r(\eta)$.

Definition 2.5. The element q is termed *minimal η-admissible over* r if q is η-admissible and if $r < s \leq q$, where s is η-admissible, implies that $q = s$.

Definition 2.6. Elements a_i, for $i = 0, 1, 2, \ldots$, are said to form an *ascending Loewy η-chain* if $a_o = 0$ and for $i > 0$, every a_i is the sum of elements which are minimal η-admissible over a_{i-1}.

Theorem 2.6. The endomorphism η of p is a uniformly splitting endomorphism if there exists an ascending Loewy η-chain of parts of p whose sum is p.

Theorem 2.7. If η is an endomorphism of p, if $p = \Sigma_i z_i$ (the summation extends from one to infinity) where $z_i \leq z_{i+1}$ and $z_i\eta \leq z_i$ for all i, and if η induces uniformly splitting

endomorphisms in every z_i, then η splits p uniformly.

III. Theory Of Direct Decompositions

If $p = a \oplus b$ and if $x \leqq p$, there exist uniquely determined mappings α and β which we define by the following equations:

$$x\alpha = (b + x)a \quad \text{and} \quad x\beta = (a + x)b.$$

The following properties are possessed by α and β:

1) α and β are endomorphisms of p.
2) $\alpha^2 = \alpha$ and $\beta^2 = \beta$.
3) $\alpha\beta = \beta\alpha = 0$.
4) $k(\alpha) = p\beta = b$ and $k(\beta) = p\alpha = a$.

The pair of endomorphisms α, β is termed a *pair of complementary decomposition endomorphisms.*

Splitting Hypothesis. If δ, ϵ is a pair of complementary decomposition endomorphisms of p and if α is a decomposition endomorphism of p, then $\alpha\delta\alpha\epsilon\alpha$ is a uniformly splitting endomorphism of p.

Theorem 3.1. (Special Refinement Theorem). If the splitting hypothesis is satisfied by p and if $p = a \oplus b = d \oplus e$, then there exist direct decompositions

$$a = a' \oplus a'', \quad b = b' \oplus b'', \quad d = d' \oplus d'', \quad \text{and} \quad e = e' \oplus e''$$

such that

$$a' \oplus b' = b' \oplus d' = d' \oplus e' = e' \oplus a' \quad \text{and}$$
$$a'' \oplus d'' = d'' \oplus e'' = e'' \oplus b'' = b'' \oplus a''.$$

Definition 3.1. The permutation $i \rightarrow i^*$ of the integers from 1 to n is said to effect an *exchange isomorphism* of the direct decomposition $p = a(1) \oplus \ldots \oplus a(n)$ upon the direct decomposition $p = b(1) \oplus \ldots \oplus b(n)$ of p if for every i

$$p = a(1) \oplus \ldots \oplus a(i\text{-}1) \oplus b(i^*) \oplus a(i + 1) \oplus \ldots \oplus a(n).$$

Theorem 3.2. (General Refinement Theorem). If the splitting hypothesis is satisfied by p, then any two direct decompositions of p possess exchange-isomorphic refinements.

VITA

Lois Aileen Hostinsky was born June 18, 1921, in Riley, Kansas. She attended the public schools of Riley and Manhattan, Kansas, and in 1939 was graduated from Manhattan High School as a member of the National Honor Society of Secondary Schools. Her undergraduate work was done at Kansas State College, where in her senior year she received a Phi Beta Kappa scholarship award. In 1943 she completed the requirements for the Bachelor of Science degree and was graduated with high honor. After doing full-time teaching of mathematics for one year at the University of Illinois, she became a scholar in mathematics in 1944-1945 and received the degree of Master of Arts in 1945. She was a fellow in mathematics in 1945-1946 and 1947-1948 and a graduate assistant in 1946-1947 and 1948-1949. She is a member of Sigma Xi, Phi Kappa Phi, Sigma Delta Epsilon, Pi Mu Epsilon, and the American Mathematical Society.